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Level spacing distribution of critical random matrix ensembles

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We consider unitary invariant random matrix ensembles that obey spectral statistics different from the Wigner-Dyson statistics, including unitary ensembles with slowly ($\sim \log^2 x$) growing potentials and the finite-temperature Fermi gas model. If the deformation parameters in these matrix ensembles are small, the asymptotically translational-invariant region in the spectral bulk is universally governed by a one-parameter generalization of the sine kernel. We provide an analytic expression for the distribution of the eigenvalue spacings of this universal asymptotic kernel, which is a hybrid of the Wigner-Dyson and the Poisson distributions, by determining the Fredholm determinant of the universal kernel in terms of a Painlevé VI transcendental function. [S1063-651X(98)51012-4]

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A main goal of quantum chaos study is to describe quantitatively statistical behaviors of spectra of classically nonintegrable systems, such as complex nuclei [1], billiards [2], QCD [3], and disordered systems [4]. A characteristic observable in such studies, used analytically or numerically to measure the deviation from integrability, is the probability $E(s)$ of having no energy levels in an interval of width s, or the distribution of spacings between adjacent levels $P(s)=E''(s)$. These observables capture the behavior of local correlations of a large number of energy levels, as the former consists of an infinite sum of integrals of regulated spectral correlators (the subscript reg denotes its regular part, i.e., with δ-functional peaks at coincident x_i's subtracted),

$$E(s)=\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int_{-s/2}^{s/2}dx_1\cdots dx_p\langle\rho(x_1)\cdots\rho(x_p)\rangle_{\rm reg}. \tag{1}$$

A technical virtue of invariant random matrix models of quantum chaotic systems [5,6] is that any p-point spectral correlation function of the former can be composed from the connected two-point function as

$$\langle\rho(x_1)\cdots\rho(x_p)\rangle_{\rm reg}=\det_{1\le i,j\le p}K(x_i,x_j). \tag{2}$$

This can most easily be proven by the orthogonal polynomial method [5]. It allows a compact expression for the level-free probability $E(s)$ as a Fredholm determinant,

$$E(s)=\det(1-\hat{K}), \tag{3}$$

over the interval $[-s/2,s/2]$.

Jimbo *et al.* [7] have made a remarkable observation that the logarithmic derivative of the Fredholm determinant, $R(s)=-[\log E(s)]'$, of the sine kernel

$$K(x,y)=\sin\pi(x-y)/\pi(x-y), \tag{4}$$

describing the bulk correlation of the Gaussian unitary ensemble (GUE), satisfies the σ form of a Painlevé V equation,

$$\left(R'(s)+\frac{s}{2}R''(s)\right)^2+[\pi sR'(s)]^2=R'(s)[R(s)+sR'(s)]^2. \tag{5}$$

Their method is subsequently generalized by Tracy and Widom [8] to kernels of the form

$$K(x,y)=\frac{\phi(x)\psi(y)-\psi(x)\phi(y)}{x-y}, \tag{6}$$

whose component functions satisfy a set of first order linear differential equations with meromorphic coefficients. This class of kernels includes Airy [9], Bessel [10], and their multicritical generalizations [11,12], describing correlations at the edges of the spectral bands.

However, such invariant random matrix ensembles are sometimes crude idealizations of physical systems with which we are concerned, based only on the symmetries of the systems. It is *a priori* unclear that random matrix ensembles can still provide quantitative descriptions of realistic physical systems where localization of the states can occur [13]. If there exists such a random matrix ensemble, it must be a nontrivial deformation of the classical invariant random matrix ensembles, so as to violate the wide universality that the Gaussian ensembles possess [14]. One such example is a random banded matrix ensemble [15], modeling quasi-one-dimensional (1D) materials. Another example is a random Hamiltonian consisting of a Gaussian random matrix H and a diagonal real random matrix V, $H_{(\alpha)}=H+\alpha V$. In such cases spectral correlation functions generally do not allow the determinant form (2), and $E(s)$ is usually evaluated only perturbatively in s, by computing each p-point correlation function [16]. However, this is not sufficient to determine $P(s)=E''(s)$ for large s, since it typically takes the form $P(s)\sim s^a\exp(-{\rm const}\times s^b)$ (generalizations of the Wigner surmise by Brody [17] and by Berry and Robnik [18]), whose exponential damping is invisible in the small-s expansion. Alternately, by treating evolution in α as a diffusion process, the joint probability distribution can be derived [19], but the level spacing distribution is yet to be obtained. The aim of this Rapid Communication is to derive $P(s)$ from random matrix ensembles, which describes a deformation of

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the Wigner-Dyson statistics, while preserving unitary invariance at the level of a partition function.

Muttalib *et al.* [20] have introduced the q analogue of the GUE with a potential $(0<q<1)$

$$V(\lambda)=\sum_{n=1}^{\infty}\log[1+2q^{n}\cosh(2\,\mathrm{arcsinh}\,\lambda)+q^{2n}]. \qquad (7)$$

They have shown that, after unfolding the spectrum

$$\lambda\mapsto x=\int^{\lambda}\rho(\lambda)d\lambda, \qquad (8)$$

this ensemble is described by a kernel

$$K(x,y)=\frac{f(x+y)}{\sqrt{f(2x)f(2y)}}\,\frac{\theta_1[\pi(x-y),e^{-\pi^2/a}]}{\sinh a(x-y)},$$
$$f(x)\equiv\frac{\theta_4(\pi x,e^{-\pi^2/a})}{\cosh ax},\quad a\equiv\frac{1}{2}\log\frac{1}{q}. \qquad (9)$$

For $e^{-\pi^2/a}\ll 1$, there exists an asymptotically translational invariant region where Eq. (9) is approximated by

$$K(x,y)=\frac{a\sin\pi(x-y)}{\pi\sinh a(x-y)}. \qquad (10)$$

It is clear from this form of the kernel that a set of eigenvalues with $|x_i-x_j|\ll 1/a$ obeys the Wigner statistics, and that with $|x_i-x_j|\gg 1/a$ obeys the Poisson statistics, i.e., is uncorrelated. Canali and Kravtsov [21] argued that the $U(N)$ symmetry is spontaneously broken in this ensemble, which induces a preferred basis in the matrices and deforms the statistics. The universality of this asymptotic kernel (10) is observed for the q-Laguerre unitary ensemble [22], and subsequently proven for ensembles with potentials that grow very slowly as [21]

$$V(\lambda)\sim(1/2a)\,(\log|\lambda|)^2 \quad (\lambda\to\infty). \qquad (11)$$

This universality can be considered as an extension of Brézin's and Zee's universality [14] of the sine kernel for polynomially increasing potentials, who proved it by deriving the asymptotic form of the wave functions

$$\psi_N(\lambda)\sim\cos\left(\pi\int^{\lambda}\rho(\lambda)d\lambda+\frac{N\pi}{2}\right), \qquad (12)$$

$$K(\lambda,\lambda')\sim\frac{\sin\left[\pi\left(\int^{\lambda}\rho-\int^{\lambda'}\rho\right)\right]}{\lambda-\lambda'}. \qquad (13)$$

For polynomially increasing potentials, the spectral density is bounded, and is locally approximated by a constant. Therefore, the unfolding is just a linear transformation, leading universally to the sine kernel (4). However, for the potential (11), the spectral density behaves as $\rho(\lambda)\sim 1/(2a\lambda)$, implying an unusual unfolding map $\lambda\mapsto x=(1/2a)\log\lambda$, while the formula (13) stays valid [23]. Then the kernel (13) universally reduces to Eq. (10) after this unfolding.

Chen and Muttalib [24] have interpreted a particular unitary ensemble with $V(x)\sim(\log x)^2$ as a fermionic system at finite temperature. This link is made more concrete by Moshe, Neuberger, and Shapiro [25], who have introduced a random matrix ensemble

$$Z=\int d^{N^2}H e^{-\mathrm{tr}H^2}\int_{U(N)}dU e^{-b\,\mathrm{tr}[U,H][U,H]^{\dagger}}, \qquad (14)$$

as yet another unitary invariant deformed ensemble. For a given unitary matrix U, the interaction $b\,\mathrm{tr}[U,H][U,H]^{\dagger}$ tries to align random Hermitian matrices H so that $[U,H]=0$. The integration over U then amounts to recovering the $U(N)$ invariance of the model that the GUE has enjoyed. The basis preference is still realized through the spontaneous breaking of the $U(N)$ invariance. After integrating over U, the above model is identical to a system of 1D free nonrelativistic fermions in a harmonic potential with curvature $m=\sqrt{1+4b}$, and at finite temperature $\beta=\mathrm{arccosh}(1+1/2b)$, extending the well-known fact that the GUE is equivalent to free harmonic fermions at zero temperature. Note that p-point correlation functions of the local spectral densities $\rho(\lambda)=\Sigma_{n=0}^{\infty}|\psi_n(\lambda)|^2/(e^{\beta(\epsilon_n-\mu)}+1)$ are thus still expressed as determinants of the kernel

$$K(\lambda,\lambda')=\sum_{n=0}^{\infty}\frac{\psi_n(\lambda)\psi_n^*(\lambda')}{e^{\beta(\epsilon_n-\mu)}+1}. \qquad (15)$$

Accordingly, its level-free probability can be expressed in terms of the Fredholm determinant of Eq. (15). Using the asymptotics of the one-particle wave function $\psi_n(x)$, given by the Hermitian polynomials, these authors have also obtained the local form of the kernel (10) with $a=\pi^2/(2N\beta)$. There, the limit $N\to\infty$ is taken while keeping the microscopic unfolded variable x fixed. The formula (10) is valid as long as $N\beta$ is not too large to invalidate the grand canonical picture. Models (11) and (14) are subsequently unified as an ensemble with multifractal eigenvectors [26], and the parameter a is identified as a measure of the multifractality.

Surprisingly, this universality within random matrix theories, if extended to orthogonal ensembles [27], is observed to encompass the 3D Anderson model, i.e., a particle hopping on the lattice with random cite energies. Canali [28] has compared his Monte Carlo results of $P(s)$ for the orthogonal ensembles with $(1/2a)\log^2 x$ potentials, with that for the Anderson Hamiltonian at the metal/insulator transition measured precisely in Ref. [29] by exact diagonalization, and observed excellent agreement by tuning the coefficient to $a\approx 2.5$. There, a is interpreted as the inverse dimensionless conductance at the transition point. Motivated by this success, we derive an analytic form of its level spacing distribution of Eq. (10) in this Rapid Communication. Although the $a\to\infty$ limit of the model does not obey the Poissonian statistics [30] as is naively expected, the error involved in the asymptotic kernel (10) is exponentially small [of order $O(e^{-\pi^2/a})$] in the above parameter range. We complete earlier attempts, which computed $P(s)$ numerically [20] or asymptotically [31].

We notice that the kernel (10) is equivalent to that of Dyson's circular unitary ensemble at finite N [5]:

$$K(x,y)=\frac{\sin\,(N/2)\,(x-y)}{N\sin\,(1/2)\,(x-y)} \qquad (16)$$

by the following analytic continuation:

$$N \to \pi i/a, \quad x \to (2a/i)x. \tag{17}$$

Tracy and Widom [8] have also proven that the diagonal resolvent kernel of Eq. (16) is determined by a second-order differential equation that is reduced to a Painlevé VI equation [32]. We will reproduce their method below.

The kernel (10) is written as

$$K(x,y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{e^{2ax} - e^{2ay}},$$
$$\phi(x) = \sqrt{\frac{2a}{\pi}}\, e^{ax}\sin\pi x, \quad \psi(x) = \sqrt{\frac{2a}{\pi}}\, e^{ax}\cos\pi x. \tag{18}$$

These component functions satisfy

$$\phi' = a\phi + \pi\psi, \quad \psi' = -\pi\phi + a\psi. \tag{19}$$

We use the bra-ket notation $\phi(x) = \langle x|\phi\rangle$, and so on [33]. Due to our choice of the component functions to be real valued (unlike [8], Sec. VD), we have $\langle x|\hat{O}|\phi\rangle = \langle\phi|\hat{O}|x\rangle$, and similarly for ψ with any self-adjoint operator $\hat{O}$ and real x. Then Eq. (18) is equivalent to

$$[e^{2a\hat{x}}, \hat{K}] = |\phi\rangle\langle\psi| - |\psi\rangle\langle\phi|, \tag{20}$$

where $\hat{x}$ and $\hat{K}$ are the multiplication operator of the independent variable and the integral operator with the kernel $K(x,y)\theta(y-t_1)\theta(t_2-y)$, respectively. Below we will not explicitly write the dependence on the end points of the underlying interval $[t_1, t_2]$. The resolvent kernel is defined as

$$R(x,y) = \left\langle x \left| \frac{\hat{K}}{1-\hat{K}} \right| y \right\rangle. \tag{21}$$

It follows from Eq. (20) that

$$\left[e^{2a\hat{x}}, \frac{\hat{K}}{1-\hat{K}} \right] = \frac{1}{1-\hat{K}} (|\phi\rangle\langle\psi| - |\psi\rangle\langle\phi|) \frac{1}{1-\hat{K}}, \tag{22}$$

that is,

$$(e^{2ax} - e^{2ay}) R(x,y) = Q(x)P(y) - P(x)Q(y),$$
$$Q(x) \equiv \langle x|(1-\hat{K})^{-1}|\phi\rangle, \quad P(x) \equiv \langle x|(1-\hat{K})^{-1}|\psi\rangle. \tag{23}$$

At a coincident point $x = y$ we have

$$2ae^{2ax}R(x,x) = Q'(x)P(x) - P'(x)Q(x). \tag{24}$$

Now, by using the identity

$$\frac{\partial\hat{K}}{\partial t_i} = (-1)^i \hat{K}|t_i\rangle\langle t_i|, \quad (i = 1,2), \tag{25}$$

we obtain

$$\frac{\partial Q(x)}{\partial t_i} = (-1)^i R(x,t_i) Q(t_i), \tag{26a}$$

$$\frac{\partial P(x)}{\partial t_i} = (-1)^i R(x,t_i) P(t_i). \tag{26b}$$

On the other hand, by using the identity (D is the derivation operator)

$$[D, \hat{K}] = \hat{K}(|t_1\rangle\langle t_1| - |t_2\rangle\langle t_2|), \tag{27}$$

which follows from the translational invariance of the kernel $(\partial_x + \partial_y)K(x-y) = 0$, we also have

$$\begin{aligned} \frac{\partial Q(x)}{\partial x} &= \langle x|D(1-\hat{K})^{-1}|\phi\rangle \\ &= \langle x|(1-\hat{K})^{-1}|\phi'\rangle \\ &\quad + \langle x|(1-\hat{K})^{-1}[D,\hat{K}](I-\hat{K})^{-1}|\phi\rangle \\ &= aQ(x) + \pi P(x) + R(x,t_1)Q(t_1) - R(x,t_2)Q(t_2), \end{aligned} \tag{28a}$$

$$\frac{\partial P(x)}{\partial x} = -\pi Q(x) + aP(x) + R(x,t_1)P(t_1) - R(x,t_2)P(t_2). \tag{28b}$$

Now we set $t_1 = -t$, $t_2 = t$, $x,y = -t$ or t, and introduce notations $\tilde{q} = Q(-t), q = Q(t)$, $\tilde{p} = P(-t), p = P(t)$, and $\tilde{R} = R(-t,t) = R(t,-t), R = R(t,t) = R(-t,-t)$. The last two equalities follow from the evenness of the kernel. Then Eqs. (23) and (24) read, after using Eq. (28),

$$\tilde{p}q - \tilde{q}p = 2\tilde{R}\sinh 2at, \tag{29a}$$

$$\tilde{p}^2 + \tilde{q}^2 = (2/\pi)(\tilde{R}^2 \sinh 2at + Rae^{-2at}), \tag{29b}$$

$$p^2 + q^2 = (2/\pi)(\tilde{R}^2 \sinh 2at + Rae^{2at}). \tag{29c}$$

The total t derivatives of Eqs. (29) lead to $(\cdot = d/dt)$

$$\tilde{p}p + \tilde{q}q = (1/\pi)(\tilde{R}\sinh 2at)^{\cdot}, \tag{30}$$

$$\dot{R} = 2\tilde{R}^2, \quad \ddot{R} = 4\tilde{R}\dot{\tilde{R}}. \tag{31}$$

The left-hand sides of Eqs. (29) and (30) satisfy an additional constraint,

$$(\tilde{p}p + \tilde{q}q)^2 + (\tilde{p}q - \tilde{q}p)^2 = (\tilde{p}^2 + \tilde{q}^2)(p^2 + q^2). \tag{32}$$

By eliminating $\tilde{p}$, p, $\tilde{q}$, q, $\tilde{R}$, and $\dot{\tilde{R}}$ from Eqs. (29)–(32), we finally obtain for $R(s)$ $(s \equiv 2t,\ ' = d/ds)$

$$\begin{aligned} &\left(a\cosh as\, R'(s) + \frac{\sinh as}{2} R''(s) \right)^2 + [\pi \sinh as\, R'(s)]^2 \\ &\quad = R'(s)([aR(s)]^2 + a\sinh 2as R(s)R'(s) \\ &\qquad + [\sinh as\, R'(s)]^2). \end{aligned} \tag{33}$$

This is our main result. It is equivalent to Eq. (5.70) of Ref. [8] after the analytic continuation (17), accompanied by a redefinition $R(s) \to (i/2a)R(s)$. (This is not trivial because Ref. [8] has used $\tilde{p} = p^*$ and $\tilde{q} = q^*$, which

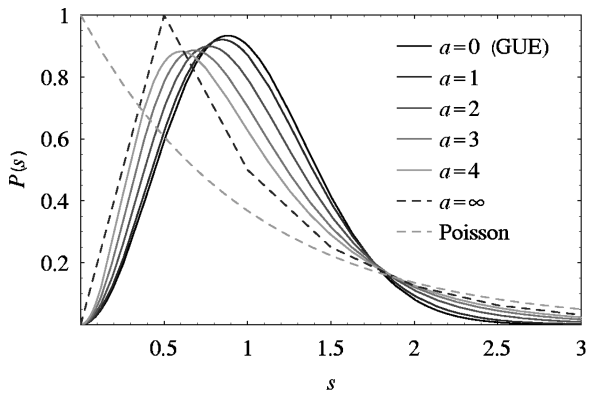

FIG. 1. Level spacing distributions $P(s)$ of the kernel (10). The limiting distribution for $a\to\infty$ [Ref. [30], Eq. (85)] and the Poisson distribution are plotted for comparison.

follow from the analytic properties of its component functions $\phi(-x)=\phi(x)^*$ and $\psi(-x)=\psi(x)^*$. It clearly reduces to the Painlevé V equation (5) for the GUE as $a\to 0$. In the Wigner-like region $as\ll 1$, we can expand hyperbolic functions into the Taylor series. Then Eqs. (1) and (2) yield

$$\begin{aligned} E(s)&=1-s+O(s^4),\\ R(s)&=-[\log E(s)]'=1+s+O(s^2). \end{aligned} \tag{34}$$

By imposing this boundary condition, we obtain a perturbative solution to Eq. (33),

$$\begin{aligned} R(s)=1+s+s^2&+\left(1-\frac{\pi^2+a^2}{9}\right)s^3+\left(1-\frac{5(\pi^2+a^2)}{36}\right)s^4\\ &+\left(1-\frac{(\pi^2+a^2)(75-4\pi^2-6a^2)}{450}\right)s^5+\cdots, \end{aligned} \tag{35}$$

$$\begin{aligned} P(s)&=[R(s)^2-R'(s)]\exp\left(-\int_0^s ds\, R(s)\right)\\ &=\frac{\pi^2+a^2}{3}s^2-\frac{(\pi^2+a^2)(2\pi^2+3a^2)}{45}s^4\\ &\quad+\frac{(\pi^2+a^2)(\pi^2+2a^2)(3\pi^2+5a^2)}{945}s^6\\ &\quad-\frac{(\pi^2+a^2)^2(\pi^2+4a^2)}{4050}s^7+\cdots, \end{aligned} \tag{36}$$

which is in accord with the expansion [Eqs. (1) and (2)].

In Fig. 1 we show plots of the level spacing distributions $P(s)$ for various a, such that $e^{-\pi^2/a}\ll 1$, obtained by numerically solving Eq. (33) under the boundary condition (34). These distributions are indeed hybrids of the rescaled Wigner-Dyson distribution $P(s)\sim s^\beta$ $(\beta=2)$ for $s\lesssim 1/a$ and the Poisson distribution $P(s)\sim e^{-\mathrm{const}\times s}$ for $s\gtrsim 1/a$. The extension of our result to the case of orthogonal ensembles $(\beta=1)$, which corresponds to the Anderson model, will be reported elsewhere [34].

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